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Equivalence Between Canonical Gibbs Measures and Stationary Measures for Stochastic Lattice-Gas Model

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It is proved that, under appropriate conditions on the jump rate and potential, one- and two-dimensional stochastic lattice-gas models (exclusion process with speed change) have only canonical Gibbs measures as their stationary measures. This extends the previously known result, which treats only a special jump rate and potential.

KEY WORDS: Canonical Gibbs measure; exclusion process; detailed balance condition; free energy.

1. INTRODUCTION

Related to the connection between ergodicity and phase transition in statistical mechanics, the relation among stationary measures and Gibbs measures has been studied for several stochastic models. It is well-known that a Gibbs (resp. canonical Gibbs) measure is a stationary measure for certain associated stochastic Ising (resp. lattice-gas) model, and assuming the translation invariance of the flip rate, potential and stationary measure, the converse statement also holds in arbitrary dimension. (cf. refs. 1 and 3).

Moreover, in one and two dimensions, without imposing translation invariance Holley and Stroock[(2)] proved that every stationary measure for a stochastic Ising model is a Gibbs measure. And by using the technique in ref. 2, Vanheuverzwijn[(5)] showed equivalence between canonical Gibbs measure and stationary measure for stochastic lattice-gas model, in one and two dimensions. However he treated only a special jump rate, so in this article we will consider more general exclusion process with speed change.

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The stochastic lattice-gas model describes jumps of individual particles from one lattice site to another with certain jump rate which depends on the energy change resulting from the jump, in such a way that the detailed balance condition is satisfied (cf. refs. 1 and 4). Notice that a stochastic Ising model has no conservation laws, while the number of particles is conserved by a lattice-gas model.

In this paper, we do not impose translation invariance of the stationary measures, jump rate and potential. So our result completes those in ref. 1. The proof heavily relies on the technique in ref. 2, and to preclude the possibility of vanishing local probabilities we will adapt the result in ref. 5.

2. NOTATION AND RESULT

The following notation mainly comes from the book of Georgii.[1]

Let $S=\mathbb{Z}^d$, $d\geqslant 1$, be the d-dimensional square lattice and take the set $\{0, 1\}^S$ as the configuration space Ω of particles on S. Let $\mathscr{S}$ denote the set of all non-empty finite subsets of S. For singletons in $\mathscr{S}$, we usually write x instead of $\{x\}$. For each $\Lambda\subset S$, $S\backslash\Lambda$ is denoted by $\tilde{\Lambda}$, and Ω_Λ, $\Omega_{\tilde{\Lambda}}$ will represent the configuration spaces over Λ and $\tilde{\Lambda}$ respectively, and let

$$X_\Lambda\colon \omega=(\omega_x)_{x\in S}\rightarrow\omega_\Lambda=(\omega_x)_{x\in\Lambda}$$

be the projection from Ω on Ω_Λ. For $\zeta\in\Omega_{\Lambda_1}$ and $\eta\in\Omega_{\Lambda_2}$ with $\Lambda_1\cap\Lambda_2=\phi$, we write $\zeta\eta$ for the joint configuration in $\Omega_{\Lambda_1}\cup\Omega_{\Lambda_2}$. Given $\omega\in\Omega$, let $N(\omega)=|\{x\in S;\omega_x=1\}|$ be the number of particles in the configuration ω. $\Omega^{n,m}$ denotes the set of configurations with at most $n-1$ sites occupied or at most $m-1$ sites vacant.

For any $\Lambda\subset S$ we denote by $\mathscr{F}_\Lambda=\sigma(X_x;x\in\Lambda)$ the σ-algebra of the events in Λ which are generated by the finite dimensional projections in Λ. It is well-known that $\mathscr{F}=\mathscr{F}_S$ holds where $\mathscr{F}$ is the Borel field defined by the product topology on Ω, and $\mathscr{E}_\Lambda=\sigma(N(X_\Lambda),\mathscr{F}_{\tilde{\Lambda}})$ denotes the σ-algebra of events which are invariant under permutation of the sites in Λ.

Let $\Phi\colon\mathscr{S}\times\Omega\rightarrow\mathbb{R}$ be a finite range potential, for which we can assume that there is a function $\mathscr{S}\rightarrow\mathbb{R}$ (again denoted by Φ) such that

$$\Phi(A,\omega)=\Phi(A)\,\omega^A\qquad A\in\mathscr{S},\omega\in\Omega$$

where

$$\omega^A=\prod_{x\in A}\omega_x=1_{\{X_A\equiv 1\}}(\omega)$$

Suppose Φ is of finite range i.e., there exists some $R_1 < \infty$ such that $\Phi(A) = 0$ unless $\operatorname{diam} A \leqslant R_1$. Then for any $\Lambda \in \mathscr{S}$, $\zeta \in \Omega_\Lambda$, $\omega \in \Omega$ the energy of ζ in Λ with boundary condition ω:

$$E_\Lambda(\zeta \mid \omega) = \sum_{\substack{A \in \mathscr{S} \\ A \cap \Lambda \neq \phi}} \Phi(A, \zeta \omega_{\tilde{\Lambda}}) \tag{2.1}$$

is well-defined and continuous as a function of ω.

Given such a potential, we say that $\mu \in \mathscr{P}(\Omega, \mathscr{F})$ is a canonical Gibbs measure with potential Φ if for all $\Lambda \in \mathscr{S}$, $\zeta \in \Omega_\Lambda$

$$\mu(X_\Lambda = \zeta \mid \mathscr{E}_\Lambda) = \frac{1}{Z_{\Lambda, N(X_\Lambda)}(\cdot)} 1_{\{\eta \in \Omega_\Lambda;\, N(\eta) = N(X_\Lambda)\}}(\zeta) \exp(-E_\Lambda(\zeta \mid \cdot)) \qquad \mu\text{-a.s.}$$

where $Z_{\Lambda, N(X_\Lambda)}(\cdot) = \sum_{\eta \in \Omega_\Lambda,\, N(\eta) = N(X_\Lambda)} \exp(-E_\Lambda(\eta \mid \cdot))$ is a normalization factor.

To define the dynamics we need the following transformations of Ω: if $\omega \in \Omega$, $x, y \in S$, we denote by ω^{xy} the configuration with

$$\omega^{xy}(z) = \begin{cases} \omega(z) & \text{if } z \neq x, y \\ \omega(y) & \text{if } z = x \\ \omega(x) & \text{if } z = y \end{cases}$$

and by ω^x the configuration with

$$\omega^x(z) = \begin{cases} \omega(z) & \text{if } z \neq x \\ 1 - \omega(x) & \text{if } z = x \end{cases}$$

Let L be the generator of the exclusion process on Ω:

$$Lf(\omega) = \sum_{\{x, y\} \subset S} c(x, y; \omega)(f(\omega^{xy}) - f(\omega))$$

where $c(x, y; \omega)$ is a nonnegative number which will be the rate at which the particles at the sites x and y interchange their positions when the total configuration is ω, therefore we suppose

$$c(x, y; \omega) = c(y, x; \omega)$$

We will assume the following conditions (a)–(e) for the jump rate $c(\cdot, \cdot; \cdot)$:

(a) finite range jump:

$$c(x, y; \cdot) = 0 \qquad \text{if } |x - y| > R_2 \text{ for some } R_2 > 0$$

(b) positive:

$$\inf_{\substack{x,\, y \in S \\ |x-y| \leqslant R_2}} \inf_{\substack{\omega \in \Omega \\ \omega_x \neq \omega_y}} c(x, y; \omega) > 0$$

(c) local:

$c(x, y; \omega)$ depends only on $\{\omega_z;\ |z-x| \leqslant R_3,\ |z-y| \leqslant R_3\}$, for some $R_3 > 0$

(d) detailed balance:

$$c(x, y; \omega) \exp(-E(xy, \omega)) = c(x, y; \omega^{xy}) \exp(-E(xy, \omega^{xy}))$$

where $E(xy, \omega) = E_{\{x, y\}}(\omega_x \omega_y \,|\, \omega)$, and to ensure the existence and uniqueness of a particle jump process with given rate,

(e)

$$\sup_{x \in S} \sum_{y \neq x} \sup_{\omega \in \Omega} c(x, y; \omega) < \infty$$

From now on, let $R = \max\{R_1, R_2, R_3\}$.

Assuming the previous conditions, the closure of the operator L generates a unique Markov semi-group on the space $C(\Omega)$ which is denoted by $\{T_t; t \geqslant 0\}$ (cf. ref. 3).

We are now in a position to state the main result of this article.

Theorem 2.1. When $d=1$ or 2, $\mu \in \mathscr{P}(\Omega, \mathscr{F})$ is a stationary measure for $\{T_t; t \geqslant 0\}$ if and only if μ is a canonical Gibbs measure with potential Φ.

Remark 2.1. Vanheuverzwijn[5] treated the case where the energy and jump rate are given by

$$E_\Lambda(\zeta \,|\, \omega) = -J \sum_{\substack{x,\, y \in \Lambda \\ |x-y|=1}} \zeta(x)\, \zeta(y) - J \sum_{\substack{x \in \Lambda,\, y \notin \Lambda \\ |x-y|=1}} \zeta(x)\, \omega(y)$$

and

$$c(x, y; \omega) = \frac{\exp(-\beta E(xy, \omega^{xy}))}{\exp(-\beta E(xy, \omega)) + \exp(-\beta E(xy, \omega^{xy}))}$$

Here $\beta \geqslant 0$ is the inverse temperature. It is easy to see that these functions satisfy our assumptions on potential and jump rate, respectively.

3. PROOF OF THEOREM 2.1

By assumptions on jump rate, Theorem 2.14 in ref. 1 shows that $\mu \in \mathscr{P}(\Omega, \mathscr{F})$ is a canonical Gibbs measure if and only if it is a reversible measure for $\{T_t; t \geqslant 0\}$, and by definition every reversible measure for $\{T_t; t \geqslant 0\}$ is a stationary measure for $\{T_t; t \geqslant 0\}$. Therefore we have only to prove that stationarity implies reversibility at least if $d=1$ or 2.

The next lemma is adapted from Lemma 3.1 in ref. 5 and its proof.

Lemma 3.1. If μ is stationary and $\mu(X_\Lambda = \eta) = 0$ for some $\Lambda \in \mathscr{S}$ and $\eta \in \Omega_\Lambda$, then there exist minimal n, m in $\mathbb{N}$ such that $\mu \in \mathscr{P}(\Omega^{n,m}, \mathscr{F})$ and $\mu(X_\Lambda = \zeta) > 0$ for all $\Lambda \in \mathscr{S}$ and all $\zeta \in \Omega_\Lambda^{n,m}$.

For each $\mu \in \mathscr{P}(\Omega, \mathscr{F})$ and $\Lambda \in \mathscr{S}$, we will introduce the following notations,

$$\Gamma_\Lambda(x, y; \zeta) = \int \mu(d\omega)\, 1_{\{X_\Lambda = \zeta\}}(\omega)\, c(x, y; \omega) \qquad \text{for} \quad x, y \in \Lambda, \zeta \in \Omega_\Lambda$$

$$\tilde{\Gamma}_\Lambda(x, y; \zeta) = \Gamma_{\Lambda \cup \{y\}}(x, y; \zeta_\Lambda \zeta_x^x) \qquad \text{for} \quad x \in \Lambda, y \in \tilde{\Lambda}, \zeta \in \Omega_\Lambda$$

and define the specific free energy of μ with respect to Φ as

$$f_\Lambda(\mu) = \int \mu(d\omega)(E_\Lambda(\omega_\Lambda \mid \alpha) + \log \mu(X_\Lambda = \omega_\Lambda))$$

where $\alpha \in \Omega$ is an arbitrarily fixed configuration.

Lemma 3.2. If μ is stationary and $\mu(X_\Lambda = \zeta) > 0$ for all $\Lambda \in \mathscr{S}$ and $\zeta \in \Omega_\Lambda$, then there exists a constant $K > 0$ independent of Λ such that

$$\begin{aligned}
&\sum_{x, y \in \Lambda}' \sum_{\zeta \in \Omega_\Lambda} (\Gamma_\Lambda(x, y; \zeta) - \Gamma_\Lambda(x, y; \zeta^{xy})) \log \frac{\Gamma_\Lambda(x, y; \zeta)}{\Gamma_\Lambda(x, y; \zeta^{xy})} \\
&\qquad + \sum_{x \in \Lambda, y \in \tilde{\Lambda}}' \sum_{\zeta \in \Omega_\Lambda} (\tilde{\Gamma}_\Lambda(x, y; \zeta) - \tilde{\Gamma}_\Lambda(x, y; \zeta^x)) \log \frac{\tilde{\Gamma}_\Lambda(x, y; \zeta)}{\tilde{\Gamma}_\Lambda(x, y; \zeta^x)} \\
&\leqslant \sum_{x \in \partial^R \Lambda, y \in \Lambda}' \sum_{\zeta \in \Omega_\Lambda} K\, |\Gamma_\Lambda(x, y; \zeta) - \Gamma_\Lambda(x, y; \zeta^{xy})| \\
&\qquad + \sum_{x \in \Lambda, y \in \tilde{\Lambda}}' \sum_{\zeta \in \Omega_\Lambda} K\, |\tilde{\Gamma}_\Lambda(x, y; \zeta) - \tilde{\Gamma}_\Lambda(x, y; \zeta^x)|
\end{aligned}$$

where

$$\partial^R \Lambda = \{z \in \Lambda;\ \mathrm{dist}(z, \tilde{\Lambda}) \leqslant R\}$$

$$\mathrm{dist}(y, \Delta) = \inf_{x \in \Delta} |x - y|$$

$$|x - y| = \max_{1 \leqslant i \leqslant d} |x_i - y_i|$$

and the primed sum indicates summation over $\{x, y\} \subset S$ with $|x - y| \leqslant R$ and each pair counted only once.

Proof. By the proof of Theorem 3.42 in ref. 1,

$$\begin{aligned}
&\left.\frac{d}{dt} f_\Lambda(\mu_t)\right|_{t=0} \\
&\quad = \sum_{\{x, y\} \cap \Lambda \neq \phi}' \int \mu(d\omega)\, c(x, y; \omega)(E_\Lambda(\omega_\Lambda^{xy} \mid \alpha) - E_\Lambda(\omega_\Lambda \mid \alpha)) \\
&\qquad + \sum_{\{x, y\} \cap \Lambda \neq \phi}' \sum_{\zeta \in \Omega_\Lambda} \log \mu(X_\Lambda = \zeta) \int \mu(d\omega)\, c(x, y; \omega)(1_{\{X_\Lambda = \zeta\}}(\omega^{xy}) \\
&\qquad - 1_{\{X_\Lambda = \zeta\}}(\omega))
\end{aligned}$$

where, given $\mu \in \mathscr{P}(\Omega, \mathscr{F})$, $\mu_t \in \mathscr{P}(\Omega, \mathscr{F})$ is defined by the relation

$$\int f\, d\mu_t = \int T_t f\, d\mu \qquad \text{for all} \quad f \in C(\Omega)$$

Now divide the summation $\sum_{\{x, y\} \cap \Lambda \neq \phi}'$ into $\sum_{x, y \in \Lambda}'$ and $\sum_{x \in \Lambda,\, y \in \tilde{\Lambda}}'$. Then by definition of Γ_Λ and $\tilde{\Gamma}_\Lambda$ we have

$$\begin{aligned}
&\left.\frac{d}{dt} f_\Lambda(\mu_t)\right|_{t=0} \\
&\quad = \sum_{x, y \in \Lambda}' \sum_{\zeta \in \Omega_\Lambda} \left(E_\Lambda(\zeta^{xy} \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\mu(X_\Lambda = \zeta^{xy})}{\mu(X_\Lambda = \zeta)} \right) \Gamma_\Lambda(x, y; \zeta) \\
&\qquad + \sum_{x \in \Lambda,\, y \in \tilde{\Lambda}}' \sum_{\zeta \in \Omega_\Lambda} \left(E_\Lambda(\zeta^{x} \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\mu(X_\Lambda = \zeta^{x})}{\mu(X_\Lambda = \zeta)} \right) \tilde{\Gamma}_\Lambda(x, y; \zeta) \\
&\quad = - \sum_{x, y \in \Lambda}' \sum_{\zeta \in \Omega_\Lambda} \left(E_\Lambda(\zeta^{xy} \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\mu(X_\Lambda = \zeta^{xy})}{\mu(X_\Lambda = \zeta)} \right) \Gamma_\Lambda(x, y; \zeta^{xy}) \\
&\qquad - \sum_{x \in \Lambda,\, y \in \tilde{\Lambda}}' \sum_{\zeta \in \Omega_\Lambda} \left(E_\Lambda(\zeta^{x} \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\mu(X_\Lambda = \zeta^{x})}{\mu(X_\Lambda = \zeta)} \right) \tilde{\Gamma}_\Lambda(x, y; \zeta^{x})
\end{aligned}$$

The second equality follows by a change of variables $\zeta \leftrightarrow \zeta^{xy} (\zeta \leftrightarrow \zeta^x)$. Next, by using the relation

$$\log \frac{\mu(X_\Lambda = \zeta^{xy})}{\mu(X_\Lambda = \zeta)} = -\log \frac{\Gamma_\Lambda(x, y; \zeta)}{\Gamma_\Lambda(x, y; \zeta^{xy})} + \log \frac{\Gamma_\Lambda(x, y; \zeta)}{\mu(X_\Lambda = \zeta)} - \log \frac{\Gamma_\Lambda(x, y; \zeta^{xy})}{\mu(X_\Lambda = \zeta^{xy})}$$

we obtain

$$\begin{aligned}
\frac{d}{dt} f_\Lambda(\mu_t)\bigg|_{t=0}
&= \frac{1}{2} \sum_{x,\, y \in \Lambda}{}' \sum_{\zeta \in \Omega_\Lambda} \left(E_\Lambda(\zeta^{xy} \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\mu(X_\Lambda = \zeta^{xy})}{\mu(X_\Lambda = \zeta)} \right) \\
&\quad \times (\Gamma_\Lambda(x, y; \zeta) - \Gamma_\Lambda(x, y; \zeta^{xy})) \\
&\quad + \frac{1}{2} \sum_{x \in \Lambda,\, y \in \tilde{\Lambda}}{}' \sum_{\zeta \in \Omega_\Lambda} \left(E_\Lambda(\zeta^x \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\mu(X_\Lambda = \zeta^x)}{\mu(X_\Lambda = \zeta)} \right) \\
&\quad \times (\tilde{\Gamma}_\Lambda(x, y; \zeta) - \tilde{\Gamma}_\Lambda(x, y; \zeta^x)) \\
&= -\frac{1}{2} \sum_{x,\, y \in \Lambda}{}' \sum_{\zeta \in \Omega_\Lambda} (\Gamma_\Lambda(x, y; \zeta) - \Gamma_\Lambda(x, y; \zeta^{xy})) \log \frac{\Gamma_\Lambda(x, y; \zeta)}{\Gamma_\Lambda(x, y; \zeta^{xy})} \\
&\quad + \frac{1}{2} \sum_{x,\, y \in \Lambda}{}' \sum_{\zeta \in \Omega_\Lambda} (\Gamma_\Lambda(x, y; \zeta) - \Gamma_\Lambda(x, y; \zeta^{xy})) \\
&\quad \times \left(E_\Lambda(\zeta^{xy} \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\Gamma_\Lambda(x, y; \zeta)}{\mu(X_\Lambda = \zeta)} - \log \frac{\Gamma_\Lambda(x, y; \zeta^{xy})}{\mu(X_\Lambda = \zeta^{xy})} \right) \\
&\quad - \frac{1}{2} \sum_{x \in \Lambda,\, y \in \tilde{\Lambda}}{}' \sum_{\zeta \in \Omega_\Lambda} (\tilde{\Gamma}_\Lambda(x, y; \zeta) - \tilde{\Gamma}_\Lambda(x, y; \zeta^x)) \log \frac{\tilde{\Gamma}_\Lambda(x, y; \zeta)}{\tilde{\Gamma}_\Lambda(x, y; \zeta^x)} \\
&\quad + \frac{1}{2} \sum_{x \in \Lambda,\, y \in \tilde{\Lambda}}{}' \sum_{\zeta \in \Omega_\Lambda} (\tilde{\Gamma}_\Lambda(x, y; \zeta) - \tilde{\Gamma}_\Lambda(x, y; \zeta^x)) \\
&\quad \times \left(E_\Lambda(\zeta^x \mid \alpha) - E_\Lambda(\zeta \mid \alpha) + \log \frac{\tilde{\Gamma}_\Lambda(x, y; \zeta)}{\mu(X_\Lambda = \zeta)} - \log \frac{\tilde{\Gamma}_\Lambda(x, y; \zeta^x)}{\mu(X_\Lambda = \zeta^x)} \right)
\end{aligned}$$

For $x, y \in \Lambda$ with $|x - y| \leqslant R$ and $\zeta \in \Omega_\Lambda$ with $\zeta_x \neq \zeta_y$, by definition of energy (2.1) and the detailed balance condition, we obtain

$$E_\Lambda(\zeta^{xy} \mid \alpha) - E_\Lambda(\zeta \mid \alpha) = \log c(x, y; \zeta^{xy} \alpha_{\tilde{\Lambda}}) - \log c(x, y; \zeta \alpha_{\tilde{\Lambda}})$$

and therefore

$$E_\Lambda(\zeta^{xy}\,|\,\alpha) - E_\Lambda(\zeta\,|\,\alpha) + \log \frac{\Gamma_\Lambda(x, y; \zeta)}{\mu(X_\Lambda = \zeta)} - \log \frac{\Gamma_\Lambda(x, y; \zeta^{xy})}{\mu(X_\Lambda = \zeta^{xy})}$$

$$= \log \frac{\int \mu(d\omega)\, 1_{\{X_\Lambda = \zeta\}}(\omega)(c(x, y; \zeta\omega_{\tilde\Lambda})/c(x, y; \zeta\alpha_{\tilde\Lambda}))}{\mu(X_\Lambda = \zeta)}$$

$$- \log \frac{\int \mu(d\omega)\, 1_{\{X_\Lambda = \zeta^{xy}\}}(\omega)(c(x, y; \zeta^{xy}\omega_{\tilde\Lambda})/c(x, y; \zeta^{xy}\alpha_{\tilde\Lambda}))}{\mu(X_\Lambda = \zeta^{xy})} \tag{3.1}$$

Now, if $x, y \in \Lambda$ with $|x - y| \leqslant R$ are not interacting with sites in $\tilde\Lambda$ i.e., $x, y \in \{z \in \Lambda; \operatorname{dist}(z, \tilde\Lambda) > R\}$, then because of assumptions on the jump rate we have

$$c(x, y; \zeta\omega_{\tilde\Lambda}) = c(x, y; \zeta\alpha_{\tilde\Lambda})$$

for all $\zeta \in \Omega_\Lambda$ and all $\omega, \alpha \in \Omega$. Hence the left-hand side of (3.1) equals to 0 in this case. Finally, let $K = \max\{K_1, K_2\}$ and

$$K_1 = \sup_{\Lambda \in \mathscr{S}} \sup_{x, y \in \Lambda} \sup_{\substack{\zeta \in \Omega_\Lambda \\ \zeta_x \neq \zeta_y}} \sup_{\alpha \in \Omega} \bigg| E_\Lambda(\zeta^{xy}\,|\,\alpha) - E_\Lambda(\zeta\,|\,\alpha) + \log \frac{\Gamma_\Lambda(x, y; \zeta)}{\mu(X_\Lambda = \zeta)} - \log \frac{\Gamma_\Lambda(x, y; \zeta^{xy})}{\mu(X_\Lambda = \zeta^{xy})} \bigg|$$

$$K_2 = \sup_{\Lambda \in \mathscr{S}} \sup_{x \in \Lambda,\, y \in \tilde\Lambda} \sup_{\zeta \in \Omega_\Lambda} \sup_{\alpha \in \Omega} \bigg| E_\Lambda(\zeta^{x}\,|\,\alpha) - E_\Lambda(\zeta\,|\,\alpha) + \log \frac{\tilde\Gamma_\Lambda(x, y; \zeta)}{\mu(X_\Lambda = \zeta)} - \log \frac{\tilde\Gamma_\Lambda(x, y; \zeta^{x})}{\mu(X_\Lambda = \zeta^{x})} \bigg|$$

then K is finite by assumptions on the potential and jump rate, thus by stationarity of μ we can complete the proof of the lemma. ∎

In a completely analogous manner we obtain the corresponding result for stationary measures on $\Omega^{n, m}$.

Lemma 3.3. If $\mu \in \mathscr{P}(\Omega^{n, m}, \mathscr{F})$ is stationary and $\mu(X_\Lambda = \zeta) > 0$ for all $\Lambda \in \mathscr{S}$ and all $\zeta \in \Omega_\Lambda^{n, m}$, then the estimate in Lemma 3.2 holds with all summations in $\zeta \in \Omega_\Lambda$ restricted to those in $\Omega_\Lambda^{n, m}$.

Lemma 3.4. When $d=1$ or 2, $\mu \in \mathscr{P}(\Omega, \mathscr{F})$ is stationary for $\{T_t; t \geqslant 0\}$ if and only if

$$\int \mu(d\omega)\, c(x, y; \omega)\, f(\omega) = \int \mu(d\omega)\, c(x, y; \omega)\, f(\omega^{xy}) \tag{3.2}$$

for all $f \in C(\Omega)$ and all $\{x, y\} \subset S$.

By Lemma 2.15 in ref. 1, (3.2) holds for all $f \in C(\Omega)$ and all $\{x, y\} \subset S$ if and only if μ is reversible for $\{T_t; t \geqslant 0\}$, so this concludes the proof of Theorem 2.1.

To prove Lemma 3.4 we need the next lemma which is taken from Lemma 1.23 in ref. 2.

Lemma 3.5. Let $\{\delta_k\}_{k=0}^{\infty}$ be a sequence of non-negative numbers with $\sum_{k=0}^{\infty} \delta_k < \infty$, and let $\{h_k\}_{k=1}^{\infty}$ be a sequence of non-negative numbers with the property that there exists a sequence of positive numbers $\{u_k\}_{k=1}^{\infty}$ such that $u_{k+1} \geqslant u_k$ for all $k \geqslant 1$ and

$$\sum_{k=1}^{n} h_k \leqslant \sum_{k=1}^{n} \delta_{n-k} h_k^{1/2} u_k^{1/2} \qquad \text{for all} \quad n \geqslant 2$$

If $\sum_{k=1}^{\infty} (1/u_k) = \infty$, then $h_k = 0$ for all $k \geqslant 1$.

Proof of Lemma 3.4. We have only to prove that stationarity of μ implies (3.2), and by Lemma 3.1 we may assume $\mu(X_\Lambda = \zeta) > 0$ for all $\Lambda \in \mathscr{S}$ and all $\zeta \in \Omega_\Lambda$. (If μ is a stationary measure as in Lemma 3.1, we have only to restrict all summations in $\zeta \in \Omega_\Lambda$ to those in $\Omega_\Lambda^{n,m}$ in the argument below.)

Define for $\Lambda \in \mathscr{S}$ and $x, y \in \Lambda$,

$$\alpha_\Lambda(x, y) = \sum_{\zeta \in \Omega_\Lambda} (\Gamma_\Lambda(x, y; \zeta) - \Gamma_\Lambda(x, y; \zeta^{xy})) \log \frac{\Gamma_\Lambda(x, y; \zeta)}{\Gamma_\Lambda(x, y; \zeta^{xy})}$$

$$\beta_\Lambda(x, y) = \sum_{\zeta \in \Omega_\Lambda} |\Gamma_\Lambda(x, y; \zeta) - \Gamma_\Lambda(x, y; \zeta^{xy})|$$

For $x \in \Lambda$ and $y \in \tilde{\Lambda}$, simple calculation shows

$$\alpha_{\Lambda \cup \{y\}}(x, y) = \sum_{\zeta \in \Omega_\Lambda} (\tilde{\Gamma}_\Lambda(x, y; \zeta) - \tilde{\Gamma}_\Lambda(x, y; \zeta^{x})) \log \frac{\tilde{\Gamma}_\Lambda(x, y; \zeta)}{\tilde{\Gamma}_\Lambda(x, y; \zeta^{x})}$$

$$\beta_{\Lambda \cup \{y\}}(x, y) = \sum_{\zeta \in \Omega_\Lambda} |\tilde{\Gamma}_\Lambda(x, y; \zeta) - \tilde{\Gamma}_\Lambda(x, y; \zeta^{x})|$$

so by Lemma 3.2 (Lemma 3.3)

$$\sum_{\substack{\{x,\, y\} \subset \mathbb{Z}^d \\ x \in \Lambda}}' \alpha_{\Lambda \cup \{y\}}(x, y) \leqslant K \sum_{\substack{\{x,\, y\} \subset \mathbb{Z}^d \\ x \in \partial^R \Lambda}}' \beta_{\Lambda \cup \{y\}}(x, y) \tag{3.3}$$

Also by (1.20) and (1.21) in ref. 2

$$\alpha_\Lambda(x, y) \leqslant \alpha_\Delta(x, y) \qquad \text{for} \quad \Lambda \subset \Delta, \quad x, y \in \Lambda \tag{3.4}$$

$$\beta_\Lambda(x, y) \leqslant C_1(\alpha_\Lambda(x, y))^{1/2} \qquad \text{for} \quad x, y \in \Lambda \tag{3.5}$$

where C_1 is a constant independent of Λ.

Let $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$ be a d-dimensional cube. By using (3.3), (3.5) and Schwarz inequality, we obtain

$$\left(\sum_{\substack{\{x,\, y\} \subset \mathbb{Z}^d \\ x \in \Lambda_n}}' \alpha_{\Lambda_n \cup \{y\}}(x, y) \right)^2 \leqslant C_2\{(2n+1)^d - ((2(n-R)+1) \vee 0)^d\} \left(\sum_{\substack{\{x,\, y\} \subset \mathbb{Z}^d \\ x \in \partial^R \Lambda_n}}' \alpha_{\Lambda_n \cup \{y\}}(x, y) \right) \tag{3.6}$$

where $\partial^R \Lambda_n = \{z \in \Lambda_n;\, \text{dist}(z, \tilde{\Lambda}_n) \leqslant R\} = \Lambda_n \backslash \Lambda_{(n-R) \vee 0}$ and $C_2 = C_2(R, d)$ is a constant independent of n.

Next, for $k \geqslant 1$ define

$$u_k = (2k+1)^d - ((2(k-R)+1) \vee 0)^d$$

$$\gamma_k = \sum_{\substack{\{x,\, y\} \subset \mathbb{Z}^d \\ x \in \partial^R \Lambda_k}}' \alpha_{\Lambda_k \cup \{y\}}(x, y)$$

$$\partial \Lambda_k = \Lambda_k \backslash \Lambda_{k-1}$$

Then

$$\begin{aligned} \sum_{k=1}^n \gamma_k &\leqslant \sum_{k=1}^n \sum_{m=(k-R) \vee 0+1}^{k} \sum_{\substack{\{x,\, y\} \subset \mathbb{Z}^d \\ x \in \partial \Lambda_m}}' \alpha_{\Lambda_n \cup \{y\}}(x, y) \\ &\leqslant R \sum_{\substack{\{x,\, y\} \subset \mathbb{Z}^d \\ x \in \Lambda_n}}' \alpha_{\Lambda_n \cup \{y\}}(x, y) \\ &\leqslant C_3 u_n^{1/2} \gamma_n^{1/2} \end{aligned}$$

where C_3 is a constant independent of n.

Now set $\delta_0 = C_3$, $\delta_k = 0$ for $k \geqslant 1$. Applying Lemma 3.5, we obtain $\gamma_k = 0$ for all $k \geqslant 1$ when $d = 1$ or 2. Then by (3.4), (3.5) and (3.6), $\beta_\Lambda(x, y) = 0$ for all $\Lambda \in \mathscr{S}$ and all $\{x, y\} \subset \Lambda$. This means that $\Gamma_\Lambda(x, y; \zeta) = \Gamma_\Lambda(x, y; \zeta^{xy})$ i.e.,

$$\int \mu(d\omega)\, 1_{\{X_\Lambda = \zeta\}}(\omega)\, c(x, y; \omega) = \int \mu(d\omega)\, 1_{\{X_\Lambda = \zeta\}}(\omega^{xy})\, c(x, y; \omega)$$

for all $\Lambda \in \mathscr{S}$, $\zeta \in \Omega_\Lambda$, $\{x, y\} \subset \Lambda$.

Finally, by the standard argument we can show (3.2) for all $f \in C(\Omega)$ and all $\{x, y\} \subset S$ and this completes the proof. ∎

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REFERENCES

1. H. O. Georgii, *Canonical Gibbs Measures*, Lect. Notes Math. **760** (Springer, 1979).
2. R. A. Holley and D. W. Stroock, In one and two dimensions, every stationary measure for a stochastic Ising model is a Gibbs state, *Commun. Math. Phys.* **55**:37–45 (1977).
3. T. M. Liggett, *Interacting Particle Systems* (Springer, 1985).
4. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Berlin/Heidelberg/New York, Springer, 1991).
5. P. Vanheuverzwijn, A note on the stochastic lattice-gas model, *J. Phys. A: Math. Gen.* **14**:1149–1158 (1981).

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